FAST METHODS FOR IDEAL BALLOONING STABILITY DETERMINATION IN 3-D CONFIGURATIONS

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MOTIVATION

– Available ballooning codes are TOO SLOW to allow direct integration into VMEC optimization procedure.
– Stability analysis is done AFTER optimization process.

AIM

– Improve and speed-up existing algorithms.
– Develop faster new approaches.
– Speed-up converter form VMEC to Boozer coordinates.

BALLOONING ALGORITHM CLASSIFICATION

ornl

INTEGRATORS

- Integrate the ballooning equation
- Get “exact” eigenvalue

VARIATIONAL METHODS

- Minimize related minimum principle over a given set of test functions
- Approximate eigenvalue from below

SPEED-UP KEYS

- MINIMIZE number of evaluations of ballooning coefficients
- Solve only for MOST USTABLE modes
- Use SYMMETRY whenever possible

IDEAL BALLOONING EQUATION

$$\frac{d}{dy}\left(P(y)\frac{dF}{dy}\right) + Q(y)F + \lambda R(y)F = 0$$

$$F \quad L^2(-\infty, +\infty)$$

$$\lambda_1 < \lambda_2, \quad < +\infty$$

$$\lambda \equiv -\gamma^2$$

- Since P(y),R(Y)>0, eigenvalues are **<u>discrete, real and ordered</u>**.
- A mode is **unstable** if an integrable F can be found for any **<u>negative</u>** value of λ

SHOOTING METHODS

- Advance F from the ends to the origin (or to one end to the origin, if symmetry) for given value of λ
- Discretization: **semi-implicit** or **implicit**
- Grid step size: **fixed** or **adaptative**
- Vary λ until continuity (or zero, if symmetry) of first derivative at the origin
- Need to be coupled to **root-finder**

SPPED-UP KEYS

•Chose most efficient root-finder algorithm

•Usually fixed grid is fastest choice

•Use solution properties to determine necessity of evaluation

TERPSICHORE

- Shooting method coupled to a root-finder using **bisection**
- Discretization: **semi-implicit**
- Grid: **fixed**
- Smallest λ **always** evaluated
- No symmetry.

MATRIX METHODS

- Discretize ballooning equation using 2nd-order scheme of fixed grid of N points and step size h
- Rewrite discretized equations in matrix form:

$$A\mathbf{F}=\lambda B\mathbf{F}$$

  where **F** is a N-length vector containing the values of F over the grid; A and B are respectively NxN tridiagonal and diagonal symmetric real matrices
- Solve <u>tridiagonal eigenvalue</u> problem:

$$|B^{-1}A-\lambda I|=0$$

SPEED-UP KEYS

- **Choose most efficient matrix eigenvalue algorithm**
- **Use positiveness of cross-diagonal products**

CONVERGENCE OF EIGENVALUE

- Respect to **grid step size** (h)

⇨ **USE Richardson's approach to the limit**

  - Solve for decreasing $\{h_k\}$ to get $\{\lambda_k\}$
  - Interpolate $\{\lambda_k\}$ to h^2-polynomial
  - Extrapolate λ to h=0 and estimate error
  - Further decrease step size if error is not small enough

- Respect to **integration box size** (xmax)

⇨ **COUPLE to Mercier's criterion**

SPEED-UP KEYS

•Reduce step size using $h_{k+1}=h_k/2$ so that only mid-points need to be evaluated at next step

BALLOONING MINIMUM PRINCIPLE

$$\lambda\left(F, F^{*}\right) \equiv \frac{\int_{-\infty}^{+\infty} \left[P(y)\left|\frac{dF}{dy}\right|^{2} - Q(y)|F|^{2} \right] dy}{\int_{-\infty}^{+\infty} R(y)|F|^{2}\, dy}$$

$$F \quad L^{2}\left(-\infty, +\infty\right)$$

- The **lowest eigenvalue** λ_1 corresponds to the **<u>global minimum</u>** of the functional $\lambda(F,F^{*})$ over all integrable functions

HERMITE VARIATIONAL METHOD

- Minimize $\lambda(F,F^*)$ using test functions which are a sum of **M Hermite polynomials weigthed by a gaussian of width** α.
- Minimization with respect to the coefficients a_i of the Hermite polynomials is done analytically resulting in a generalized eigenvalue matrix problem

$$C(\alpha)\mathbf{A}=\lambda(\alpha)\,D(\alpha)\mathbf{A}$$

  where **A** is a M-length vector containing the coefficients a_i and $C(\alpha)$ and $D(\alpha)$ are MxM symmetric matrices.
- Elements of C and D are integrals over the whole domain of products of the test function with ballooning coefficients
- Minimization with respect to α is done using a 1-D algorithm

SPEED-UP KEYS

- **Use Richardson's scheme to evaluate integrals**
- **Choose efficient eigenvalue and minimization algorithms**

VGAUSS: VARIATIONAL ROUTINE

- Uses as test function a sum of **three arbitrary centered gaussians**
- Integration is done with a **fixed step rule**
- Minimization is then carried out over **SEVEN** parameters using simplex method.

ALGORITHM PERFORMANCE I

VGAUSS: 3 gaussians+simplex method

TERPSICHORE: Fixed grid+bisection

VHERM3: 3 lowest order Hermite polynomials+Brent 1-D minimizer+Richardson's scheme

MATRIX: using Richardson's scheme

FSHOOT: Fixed grid+VWDB root-finder+ Richardson's scheme

ASHOOT: Adaptative grid + VWDB root-finder

ALGORITHM PERFORMANCE II

ALGORITM	CPU-TIME(s/s)	EVALUATIONS	RATIO TO TERP
VGAUSS	~60	270,000	3.0
TERPS	~2	90,000	1.0
VHERM3	~0.5	18,435	0.2
MATRIX	~0.5	23,895	0.26
FSHOOT	~0.5	22,425	0.25
ASHOOT	~3.5	173,760	1.9

- Speed depends very strongly on **number of evaluations**.
- TERPSICHORE can be speeded up reducing number of points but this **is not done automatically**. And different surfaces require <u>different</u> number of points.
- An **extra 5x factor** can be gained from optimizing Boozer conversion algorithm.

CONCLUSIONS

- Ballooning stability determination can be speeded up **minimizing the number of evaluations** of the ballooning coefficients along the field line using **Richardson's scheme**
- At the same time, control of the **convergence of the eigenvalue** is ensured
- Further speed can be gained using: a) **more efficient** algorithms, b) **symmetry** and c) **fast stability criteria** to decide which eigenvalues to evaluate
- Preliminary results show an improvement in speed of about **4(8) times with respect to TERPSICHORE** and more than **100 times** with respect to previous variational routines.